MATH 507a **QUALIFYING EXAM** **August 2017.**

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

When the problem asks you to compute something, you are expected to simplify the answer as much as possible.

1. (10 points) Let $\{A_n,\ n \geq 1\}$ be independent events and let 1_{A_k} denote the indicator of the event A_k. Suppose that

$$\sum_{n=1}^{\infty} P(A_n) = +\infty.$$

(a) Show that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} 1_{A_k}}{\sum_{k=1}^{n} P(A_k)} = 1$$

in probability.

(b) Show that the convergence in part (a) is with probability one. One approach is to first consider convergence along a suitable subsequence $\{n_j\}$.

2. (10 points) Let $U_1, U_2, \ldots$ be iid random variables, each uniform on $[0, 10]$, and define $X_n = \prod_{k=1}^{n} U_k$.

(a) Compute $\mu_n = \mathbb{E}X_n$.

(b) Show that $\lim_{n\to\infty} \big(X_n\big)^{1/n}$ exists in distribution, and identify the distribution.

(c) Denote by m_n the median of X_n. Show that $\lim_{n\to\infty} \frac{m_n}{\mu_n} = 0$.

3. (10 points) Let $X_1, X_2, \ldots$ be non-negative iid random variables with finite second moment.

(a) Show that

$$\lim_{n\to\infty} \frac{\max\limits_{1\leq k\leq n} X_k}{\sqrt{n}} = 0$$

with probability one.

(b) Use part (a) to show that $\lim_{n\to\infty} n^{-3/2} \sum_{k=1}^{n} X_k^3 = 0$ with probability one.

4. (5 points) Let X and Y be two random variables on the same probability space, and let φ_X, φ_Y be their characteristic functions. Show that

$$\sup_{-\infty<t<+\infty} \big|\varphi_X(t) - \varphi_Y(t)\big| \leq 2P(X \neq Y).$$