Numerical Analysis Screening Exam, Fall 2017

First Name:

Last Name:

There are a total of 4 problems on this test. Please start each problem on a new page and mark clearly the problem number and the page number on top of each page. Please also write your names on top of the pages.

I. Direct Methods (25 Points)

Suppose A and B are non-singular $n \times n$ matrices and the $2n \times 2n$ matrix

$$C = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

is also non-singular.

(a) Show that

$$\det(C) = \det(A)\det(A - BA^{-1}B)$$

(b) Show that both $A + B$ and $A - B$ are non-singular.

(c) Consider the system of equations $Cx = b$. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where x_1, x_2, b_1, b_2 are in $\mathbb{R}^n$. Let y_1 and y_2 be the solutions to

$$(A + B)y_1 = b_1 + b_2 \qquad \text{and} \qquad (A - B)y_2 = b_1 - b_2$$

Show that

$$x_1 = \frac{1}{2}(y_1 + y_2)$$
$$x_2 = \frac{1}{2}(y_1 - y_2)$$

(d) What is the numerical advantage of finding the solution to $Cx = b$ in this way?

II. Least Squares Problem (25 Points)

Consider the following least square optimization problem

$$\min_{x\in\mathbb{R}^3} \|Ax-b\|_2^2, \quad \text{where} \quad A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}.$$

(a) Show that $x = (3,1,0)^T$ is a solution to the least square problem.

(b) Find the minimum norm solution for this problem.

(c) Consider any vector $b = (b_1, b_2, b_3)^T$, show that a solution for the least square problem

$$\min_{x\in\mathbb{R}^3} \left\| \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 4 \\ 1 & 1 & 2 \end{pmatrix} x - b \right\|_2^2.$$

is given by

$$x = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \text{where} \quad \alpha = \frac{b_1+b_3+2b_2}{4}, \quad \beta = \frac{b_1+b_3-2b_2}{4}.$$

(d) Using an approach similar to (b) to find the pseudo inverse of the matrix A.

III. Eigenvalue Problems (25 Points)

Consider the following matrix

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 3 & -3 & 3 \\ 1 & 3 & -1 \end{pmatrix}.$$

(a) Show that $x = (1, 1, 1)^T$ is an eigenvector of A.

(b) Consider matrix Q given by

$$Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}.$$

Show that $\hat{A} = Q^* A Q$ has same eigenvalues as matrix A and

$$\hat{A} = \begin{pmatrix} 3 & -2/\sqrt{6} & \sqrt{2} \\ 0 & -3 & \sqrt{3} \\ 0 & -1/\sqrt{3} & -5 \end{pmatrix}.$$

(c) Use the result of (b) to find a Schur decomposition of A .

(d) Try to perform one step of the LR algorithm on A. What transformation may be needed to compute the eigenvalues of matrix A using the LR algorithm?

IV. Iterative Methods (25 Points)

We want to solve the system

$$Ax = b$$

for an invertible matrix $A \in \mathbb{R}^{n\times n}$. Assume that we have a computational algorithm that computes an approximation $x_1 = \overline{A^{-1}}b$ of x. Ideally, we would choose $\overline{A^{-1}} = A^{-1}$ but let's say for example due to roundoff errors this is not available. In order to improve the result x_1 we can use the following strategy: For $\Delta := x - x_1$ we have

$$A\Delta = b - Ax_1.$$

Again we cannot compute this exactly but we can apply our algorithm $\overline{A^{-1}}$ to obtain $\bar{\Delta} = \overline{A^{-1}}(b - Ax_1)$. Then

$$x_2 = x_1 + \bar{\Delta} \approx x_1 + \Delta = x$$

is hopefully a better approximation of x. We can now iteratively apply this strategy to obtain a sequence x_k, $k = 1, 2, \ldots$.

(a) Rewrite this process as an iterative solver of the form $x_{n+1} = Bx_n + c$ for the system $Ax = b$.

(b) Show that for $\|\overline{A^{-1}} - A^{-1}\| < \|A\|$ this method converges to the correct solution x.