1. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample from the uniform distribution on a disc $X^2 + Y^2 \leq \theta^2$, where $\theta > 0$ is unknown. That is, the probability density function of (X, Y) is

$$f_{(X,Y)}(x, y; \theta) = \frac{1}{\pi\theta^2} 1_{[0,\theta]}(\sqrt{x^2 + y^2})$$

   (a) Find a complete sufficient statistic of θ and its distribution.

   (b) Find the UMVU estimator of θ.

   (c) Find the maximum likelihood estimator of θ.

2. Let $Y_1, \ldots, Y_n$ be independent with $Y_i \sim N(\alpha x_i + \beta \log x_i, \sigma^2)$, where $x_1, \ldots, x_n$ are given positive constants, not all equal, and α, β, σ are unknown parameters.

   (a) Prove that the MLE of β is

$$\widehat{\beta} = \frac{S_{ly}S_{x^2} - S_{lx}S_{xy}}{S_{x^2}S_{l^2} - S_{lx}^2}, \tag{1}$$

   where

$$S_{ly} = \sum_i (\log x_i) Y_i, \quad S_{x^2} = \sum_i x_i^2, \quad S_{l^2} = \sum_i (\log x_i)^2, \quad \text{etc.}$$

   (b) Find the distribution of $\widehat{\beta}$, including giving any parameter values for this distribution. Is $\widehat{\beta}$ unbiased for β? Justify your answers.

   (c) Suppose now that you may choose the values of the x_i, but each one must be either 1 or 10. How many of the n observations should you choose to make at $x_i = 1$ in order to minimize the variance of the resulting $\widehat{\beta}$? You can assume that n is a fixed multiple of 11.