REAL ANALYSIS GRADUATE EXAM

Fall 2015

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form p/q such that $q \geq 2$ and $|x - p/q| < 1/(q \log q)^2$. (Hint: Consider intervals of length $2/(q \log q)^2$ centered at rational points p/q.)

2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

3. Let (X, μ) be a finite measure space. Assume that a sequence of integrable functions f_n satisfies $f_n \to f$ in measure, where f is measurable. Assume that f_n satisfies the following property: For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \leq \delta \implies \int_E |f_n| \, d\mu \leq \epsilon.$$

Prove that f is integrable and that

$$\lim_n \int_X |f_n - f| \, d\mu = 0.$$

4. Consider the following two statements about a function $f\colon [0,1] \to \mathbb{R}$:
(i) f is continuous almost everywhere
(ii) f is equal to a continuous function g almost everywhere.
Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counterexample.