MATH 507a GRADUATE EXAM

FALL 2015

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $X_1, X_2, ...$ are iid with values in a bounded interval $[a, b]$ with density $f(x)$.

(a) Suppose the distribution is uniform in $[a, b]$. Show that $\liminf_n n(X_n - a) = 0$ a.s.

(b) Still supposing the distribution is uniform, what can you say about $\liminf_n n^2(X_n - a)$?

(c) Suppose instead that the endpoint $a = 0$ and the density satisfies $f(x) \sim cx^{-1/2}$ as $x \to 0$, with $0 < c < \infty$. (Here $\sim$ means the ratio converges to 1.) Find the new values of the lim infs in (a) and (b).

(2) Let ξ be a non-negative random variable. Prove that for $b > 0$,

$$\lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E} e^{-t\xi} = -b$$

if and only if

$$\mathbb{P}(\xi \geq b) = 1.$$

(3) Suppose that $0 < \alpha \leq 2$, that $X, X_1, X_2, \ldots$ are independent and identically distributed, and that $S_n = X_1 + \cdots + X_n$. Assume that X has characteristic function

$$\phi(u) := \mathbb{E}(e^{iuX}) = e^{-|u|^\alpha}.$$

(a) Is X symmetric (meaning X and $-X$ have the same distribution)?

(b) Find a function $f(n)$, depending on α, such that $S_n/f(n)$ has the same distribution as X.

(c) For what values of α does S_n satisfy the conclusion of the central limit theorem, that is, $S_n/\sqrt{n}$ converges in distrubution to a normal random variable (with strictly positive variance) as $n \to \infty$?

(d) For what values of α does S_n/n converge to 0 in probability as $n \to \infty$?