Numerical Analysis Screening Exam, Fall 2015

First Name:

Last name:

Student ID Number:

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Problem 1 Consider the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \epsilon & \epsilon \\ 0 & \epsilon & \epsilon + \epsilon^2 \end{pmatrix}$$

(a) Use the Gram-Schmidt orthogonalization method on the columns of matrix A to derive a QR-decomposition of A.

(b) Use the Householder QR-factorization method to perform the same task as (a).

(c) Based on your calculations in (a) and (b), determine which method would lead to a more accurate factorization on an actual computer and justify your reasoning.

(d) A graduate student programmed both techniques in Matlab and tested them for the case $\epsilon = 10^{-10}$. He evaluated the norm of $A - Q * R$ in Matlab and found that the norm was equal to 0 for the Gran-Schmidt factorization and 4.6032×10^{-26} for the Householder factorization. Is this consistent with your conclusion in (c)? What other quantities would you suggest him to examine that may support your conclusion in (c)?

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Problem 2. Let A be an $n \times n$ real-valued, symmetric positive definite matrix and $b \in \mathbb{R}^n$. Consider the following two-stage iterative procedure for solving the system of equations $Ax = b$:

$$\begin{aligned} x_{n+\frac{1}{2}} &= x_n + \omega_1(b - Ax_n) \\ x_{n+1} &= x_{n+\frac{1}{2}} + \omega_2(b - Ax_{n+\frac{1}{2}}) \end{aligned}$$

(a) Let $e_n = x - x_n$ be the error between the n-th iterate x_n and the exact solution x. Find the matrix K such that $e_{n+1} = Ke_n$.

(b) Find the eigenvalues of K in terms of the eigenvalues of A, ω_1, and ω_2.

(c) Show that ω_1 and ω_2 can be chosen so that the method converges with any initial condition. Express the rate of convergence in terms of λ_M and λ_m which correspond to the largest and smallest eigenvalues of A, respectively.

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PROBLEM 3. Consider a least square minimization problem:

$$\text{minimize}\|Ax - b\|_2^2, \qquad x \in \mathbb{R}^n, b \in \mathbb{R}^m, \tag{1}$$

and a regularized form of the (1):

$$\text{minimize}\|Ax - b\|_2^2 + \alpha\|x\|_2^2, \qquad \alpha > 0. \tag{2}$$

(a) State and justify a necessary and sufficient condition for a vector x_0 to be a solution of (1) and determine whether or not this problem always has a unique solution?

(b) State and justify a necessary and sufficient condition for a vector x_0 to be a solution of (2) and determine whether or not this problem always has a unique solution?

(c) Let $\mathcal{R}(A^T)$ be the range of A^T and let $\mathcal{N}(A)$ be the null space of A. Explain why a solution of (2) must be in $\mathcal{R}(A^T)$.

(d) Suggest a method for approximating a minimal norm solution of (1) using a sequence of solutions of (2) and justify your answer.

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PROBLEM 4. Let A be an $n \times n$ skew-Hermitian (i.e. $A^H = -A$) matrix.

(a) Show that $I + A$ is invertible.

(b) Show that $U = (I + A)^{-1}(I - A)$ is unitary.

(c) Show that if U is unitary with $-1 \notin \sigma(U)$, then there exists a skew-Hermetian matrix A such that $U = (I + A)^{-1}(I - A)$.

(d) Show that if B is an $n \times n$ normal matrix (i.e. $B^H B = BB^H$) then it is unitarily similar to a diagonal matrix.

(e) Let C be an $n \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Show that if λ is an eigenvalue of C, then $|\lambda| \leq \sigma_1$ and that $|\det(C)| = \prod_{i=1}^{n} \sigma_i$.

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