Geometry and Topology Graduate Exam

Fall 2015

Problem 1. (15 points)

(a) Define the two notions "homotopy between two maps" and "homotopy equivalences between two spaces".

(b) Give an example of two topological spaces X and Y that are homotopy equivalent but are not homeomorphic.

(c) Give an example of path-connected topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.

(d) Give an example of path-connected topological spaces X and Y that have isomorphic first homology groups $H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ but whose fundamental groups are not isomorphic.

Problem 2. (15 points) Let T be the 2–dimensional torus, and let K be the Klein bottle.

(a) Describe a twofold covering map $p\colon T \to K$. ("Twofold" means that the preimage of each point of K consists of two points of T.)

(b) Pick base points $x_0 \in T$ and $y_0 \in K$ such that $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T;x_0)$ and $\pi_1(K;y_0)$ and, for each generator of $\pi_1(T;x_0)$, express its image under the induced homomorphism $p_*\colon \pi_1(T;x_0) \to \pi_1(K;y_0)$ in terms of the generators of $\pi_1(K;y_0)$.

Problem 3. (25 points) Let Σ_g and $\Sigma_{g'}$ be closed orientable surfaces of genus g and $g' > 0$, respectively. Let $f\colon B^2 \to \Sigma_g$ be an embedding of the 2–dimensional disk B^2, and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$. Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f'\colon B^2 \to \Sigma_{g'}$. Finally, let X be the topological space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ'; namely, X is obtained from the disjoint union $\Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for every $x \in S^1$.

(a) Compute the fundamental group of X.

(b) Compute all homology groups of X.

(c) Is X homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

Problem 4. (15 points) Let M be a manifold of dimension n, and let ω be a differential form of degree $n-1$ on M. Suppose that $\int_N \omega = 0$ for every $(n-1)$–dimensional oriented closed submanifold N of M. Show that $dw = 0$. (Possible hint: look at small spheres.)

Problem 5. (15 points) Consider the vector fields $\mathbf{v} = \partial_x + xz\partial_z$ and $\mathbf{w} = \partial_y + yz\partial_z$ in $\mathbb{R}^3$. If P is a point of $\mathbb{R}^3$, does there exist a local coordinate system in a neighborhood of P in which $\mathbf{v}$ and $\mathbf{w}$? Namely, is there a diffeomorphism $\phi\colon U \to V$ from a neighborhood U of P to an open subset $V \subset \mathbb{R}^3$ that sends $\mathbf{v}$ to ∂_x and $\mathbf{w}$ to ∂_y?

Problem 6. (15 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of one complex variable. Recall that the one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ is homeomorphic to the sphere S^2.

(a) Show that f extends to a continuous map $\bar{f}\colon S^2 \to S^2$.

(b) Show that the degree of $\bar{f}$ (in the sense of topology or geometry) is equal to the degree of the polynomial f (in the algebraic sense).