DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2015

The exam consists of seven problems. The first three problems are related to ODEs while the last four are related to PDEs. Good luck!

1. Consider the following system of ODE's

$$\begin{aligned} x' &= x + 2y - x^2 - y^2 + x^3 y - x^3 \\ y' &= 3x + 3y + xy^2 - xy \end{aligned}$$

   Let C be a small circle centered at the origin. What is the measure of the set of initial conditions on C that give rise to solutions $(x(t), y(t))$ with the property that

$$\lim_{t\to\infty} (x(t), y(t)) = (0, 0)?$$

   Assume $\mu(C) = 1$.

2. For any $x_0 \in \mathbb{R}$ show that the solution of

$$\frac{dx}{dt} = \frac{1 + x^2}{2 + x + 3x^2}, \qquad x(0) = x_0$$

   can be continued to all $t \in \mathbb{R}$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function with $f(0) \neq 0$ and A an $n \times n$ matrix with $\det A \neq 0$. Consider the system of differential equations

$$\frac{dx}{dt} = Ax + \mu f(x), \qquad \text{ODE}$$

   where $\mu \in [0, \mu_0)$ is a small parameter. Show that for $\mu_0 > 0$ sufficiently small there exists a C^1 family of stationary points $\xi(\mu)$ of ODE such that $\xi(0) = 0$.

4. Let $\Omega = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}$ and assume that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $f(0) = 0$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of the following problem

$$\Delta u = f(u) \quad (x, y) \in \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

   Show that u is identically zero.

5. Prove that for any function $f \in H^1(0, \pi)$ the following inequality holds

$$\int_0^\pi f^2\,dx \leq \int_0^\pi (f')^2\,dx + \left(\int_0^\pi f\,dx\right)^2.$$

6. Consider the 3-dimensional wave equation on a half space

$$u_{tt} = \Delta u \qquad x \in \Omega = \{(x_1, x_2, x_3);\ x_1 > 0\},$$

with the initial and boundary conditions

$$u(0,x) = g(x), \quad u_t(0,x) = 0; \qquad u(t,x) = 0 \text{ for } x \in \partial\Omega.$$

Assume that g is a smooth function with bounded support, which vanishes on $\partial\Omega = \{(x_1, x_2, x_3);\ x_1 = 0\}$,

(i) Write an explicit formula for the solution $u = u(x,t)$.

(ii) Prove that $\lim_{t\to\infty} ||u(t,\cdot)||_{L^\infty} = 0$.

(iii) Prove that the total energy

$$E = \frac{1}{2}\int_\Omega \left(u_t^2 + |\nabla u|^2\right) dx$$

is constant in time.

7. Write a brief discussion of the method of characteristics and then demonstrate the method by solving

$$u_x u_y = u \quad \text{in } \Omega \qquad u = y^2 \text{ on } \partial\Omega.$$

where $\Omega = \{(x,y)\ ;\ x > 0\} \subset \mathbb{R}^2$.