MATH 505a GRADUATE EXAM

FALL 2015

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1)(a) There are $n = 100$ balls, labelled 1 to n, and these are thrown into n boxes, also labelled 1 to n; all n^n outcomes are equally likely. Whenever ball i lands in box j, and $|i - j| \leq 1$, a point is scored, so the total score, call it X, can take on any value from 0 to n. Note: ball 1 in box n scores nothing, and ball n in box 1 scores nothing. Compute exactly, and simplify, as either an expression in n, or a decimal: $E(X)$ and $\mathrm{Var}(X)$.

(b) Pick the closest approximation: in part (a), $\mathrm{P}(X = 0)$ is close to 1, 1/3, 1/20, 1/100. You may reason informally, but you must describe your reasoning, which should involve the approximate distribution of X.

(c) Change the story in (a) to: there are $n = 100$ cards, numbered 1 to n, and they are placed in positions 1 to n around a circle at random, so that all $n!$ outcomes are equally likely. Whenver card i is placed in position i or $i + 1$, a point is scored, so the total score, call it Y, can take on any value from 0 to n. Here for card n, since the cards are in a circle, "position $n+1$" should be interpreted as position 1. Compute exactly, and simplify, as either an expression in n, or a decimal: $E(Y)$ and $\mathrm{Var}(Y)$.

(2)(a) A coin comes up heads with probability $p \in [0, 1]$; it is tossed independently n times, and X is the total number of heads. Simplify the generating function

$$G(s) = E(s^X),$$

and show how derivatives of G can be used to calculate $E(X)$ and $E(X^2)$.

(b) Random variables $U, U_1, U_2, \ldots, U_n$ are independent, and uniformly distributed in [0,1]. Let

$$Y = \sum_{1}^{n} 1_{\{U_i < U\}}$$

be the sum of indicators, counting how many of the U_i are less than U. Simplify the generating function

$$H(s) = E(s^Y),$$

and then simplify the ratio, $\mathrm{P}(Y = 2)/\mathrm{P}(Y = 1)$. HINT: conditionally on $U = p$, this Y is distributed as the X in part (a).

(3) Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$ be iid uniform on $(0, 1)$.

(a) Compute the probability density function of $\max(Y_1, \ldots, Y_m)$.

(b) Compute the probability density function of

$$\frac{\max(X_1, \ldots, X_n)}{\max(Y_1, \ldots, Y_m)}.$$

HINT: One method (not the only one) is to use conditioning.