ALGEBRA QUALIFYING EXAM FALL 2015

Work all of the problems. *Justify the statements in your solutions by reference to specific results, as appropriate.* Partial credit is awarded for partial solutions. The set of integers is $\boldsymbol{Z}$, the set of rational numbers is $\boldsymbol{Q}$, and set of the complex numbers is $\boldsymbol{C}$.

Hand in the exam with problems in numerical order.

1. If M is a maximal ideal in $\boldsymbol{Q}[x_1, \ldots, x_n]$ show that there are only finitely many maximal ideals in $\boldsymbol{C}[x_1, \ldots, x_n]$ that contain M.

2. Let R be a right Noetherian ring with 1. Prove that R has a *unique* maximal nilpotent ideal $P(R)$. Argue that $R[x]$ also has a unique maximal nilpotent ideal $P(R[x])$. Show that $P(R[x]) = P(R)[x]$.

3. Up to isomorphism, describe the possible structures of any group of order 182 as a direct sum of cyclic groups, dihedral groups, other semi-direct products, symmetric groups, or matrix groups. (Note: 91 is not a prime!)

4. Let $K = \boldsymbol{C}(y)$ for an indeterminate y and let $p_1 < p_2 < \cdots < p_n$ be primes (in $\boldsymbol{Z}$). Let $f(x) = (x^{p_1} - y)\cdots(x^{p_n} - y) \in K$ with splitting field L over K

   a) Show each $x^{p_j} - y$ is irreducible over K.

   b) Describe the structure of $Gal(L/K)$.

   c) How many intermediate fields are between K and L?

5. In any finite ring R with 1 show that some element in R is not a sum of nilpotent elements. Note that in all $M_n(\boldsymbol{Z}/n\boldsymbol{Z})$ the identity matrix is a sum of nilpotent elements. (Hint: What is the trace of a nilpotent element in a matrix ring over a field?)

6. Let R be a commutative principal ideal domain.

   (1) If I and J are ideals of R, show $R/I \otimes_R R/J \cong R/(I + J)$.

   (2) If V and W are finitely generated R modules so that $V \otimes_R W = 0$, show that V and W are torsion modules whose annihilators in R are relatively prime.

7. Let $g(x) = x^{12} + 5x^6 - 2x^3 + 17 \in \boldsymbol{Q}[x]$ and F a splitting field of $g(x)$ over $\boldsymbol{Q}$. Determine if $Gal(F/\boldsymbol{Q})$ is solvable.