Fall 2014 Math 541b Exam

1. (a) Let $q_{x,y}$ be a Markov transition function, and π_x a probability distribution on a finite state space S. Show that the Markov chain that accepts moves made according to $q_{x,y}$ with probability

$$p_{x,y} = \min\left\{\frac{\pi_y q_{y,x}}{\pi_x q_{x,y}}, 1\right\},$$

and otherwise remains at x, has stationary distribution π_x. Show that if $q_{x,y}$ and π_x are positive for all $x, y \in S$ then the chain so described has unique stationary distribution π_x.

(b) Let $f(y)$ and $g(y)$ be two probability mass functions, both positive on $\mathbb{R}$. With X_1 generated according to g, consider the Markov chain $X_1, X_2, \ldots$ that for at stage $n \geq 1$ generates an independent observation Y_n from density g, and accepts this value as the new state X_{n+1} with probability

$$\min\left\{\frac{f(Y_n)g(X_n)}{f(X_n)g(Y_n)}, 1\right\}$$

and otherwise sets X_{n+1} to be X_n. Prove that the chain converges in distribution to a random variable with distribution f.

(c) The accept/reject method. Let f and g be density functions on $\mathbb{R}$ such that the support of f is a subset of the support of g, and suppose that there exists a constant M such that $f(x) \leq Mg(x)$. Consider the procedure that generates a random variable with distribution g, an independent random variable with the uniform distribution U on $[0,1]$ and sets $Y = X$ when $U \leq f(X)/Mg(X)$. Show that Y has density f.

2. Let f be a real valued function on $\mathbb{R}^n$, and $Z = f(X_1, \ldots, X_n)$ for $X_1, \ldots, X_n$ independent random variables.

(a) With $E^{(i)}(\cdot) = E(\cdot|X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ show the following version of the Efron-Stein inequality

$$\text{Var}(Z) \leq E\left(\sum_{i=1}^{n}(Z - E^{(i)}Z)^2\right). \tag{1}$$

Hint: With $E_i(\cdot) = E(\cdot|X_1, \ldots, X_i)$, show that

$$Z - EZ = \sum_{i=1}^{n} \Delta_i \quad \text{where} \quad \Delta_i = E_i Z - E_{i-1} Z,$$

compute the variance of Z in this form, use properties of conditional expectation such as $E_i(E^{(i)}(\cdot)) = E_{i-1}(\cdot)$, and (conditional) Jensens' inequality.

(b) Letting $(X_1', \ldots, X_n')$ be an independent copy of $(X_1, \ldots, X_n)$, with

$$Z_i' = f(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n),$$

show that

$$\mathrm{Var}(Z) \leq \frac{1}{2} E\left(\sum_{i=1}^{n} (Z - Z_i')^2\right).$$

Hint: Express the right hand side of (1) in terms of conditional variances, and justify and use the conditional version of the fact that if X and Y are independent and have the same distribution then the variance of X can be expresses in terms of $E(X-Y)^2$.