Fall 2014 Math 541a Exam

1. Let p, q be values in $[0, 1]$ and $\alpha \in (0, 1]$. Assume α and q known, and that p is an unknown parameter we would like to estimate. A coin is tossed n times, resulting in the sequence of zero one valued random variables $X_1, \ldots, X_n$. At each toss, independently of all other tosses, the coin has probability p of success with probability α, and probability q of success with probability $1 - \alpha$.

   (a) Write out the probability function of the observed sequence, and compute the maximum likelihood estimate $\widehat{p}$ of p, when p is considered a parameter over all of $\mathbb{R}$. Verify that when $\alpha = 1$ one recovers the standard estimator of the unknown probability.

   (b) Show $\widehat{p}$ is unbiased, and calculate its variance.

   (c) Calculate the the information bound for p, and determine if it is achieved by $\widehat{p}$.

   (d) If one of the other parameters is unknown, can p still be estimated consistently?

2. Let $\mathbf{X} \in \mathbb{R}^n$ be distributed according the density or mass function $p(\mathbf{x}; \theta)$ for $\theta \in \Theta \subset \mathbb{R}^d$.

   (a) State the definition for $T(\mathbf{X})$ to be sufficient for θ.

   (b) Prove that if the (discrete) mass functions $p(\mathbf{x}; \theta)$ can be factored as $h(\mathbf{x})g(T(\mathbf{x}), \theta)$ for some functions h and g, then $T(\mathbf{X})$ is sufficient for θ.

   (c) Let $X_1, \ldots, X_n$ be independent with the Cauchy distribution $\mathcal{C}(\theta), \theta \in \mathbb{R}$ given by

   $$p(x; \theta) = \frac{1}{\pi(1 + (x - \theta)^2)}.$$

   Prove that the unordered sample $S = \{X_1, \ldots, X_n\}$ can be determined from any $T(\mathbf{X})$ sufficient for θ.(Hint: Produce a polynomial from which S can be determined).