REAL ANALYSIS GRADUATE EXAM

Fall 2014

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Assume that f is integrable on $(0,1)$. Prove that

$$\lim_{a\to\infty}\int_0^1 f(x)\,x\,\sin(ax^2)\,dx = 0.$$

2. Let $(X,\mathcal{M},\mu)$ be a measure space, and let f and $f_1, f_2, f_3 \ldots$ be real valued measurable functions on X. If $f_n \to f$ in measure and if $F\colon \mathbb{R}\to\mathbb{R}$ is uniformly continuous, prove that $F\circ f_n \to F\circ f$ in measure.

3. Let f_n be **nonnegative** measurable functions on a measure space $(X,\mathcal{M},\mu)$ which satisfy $\int f_n\,d\mu = 1$ for all $n = 1, 2, \ldots$. Prove that

$$\limsup_n\,(f_n(x))^{1/n} \le 1$$

for μ-a.e. x.

4. Let $-\infty < a < b < \infty$. Suppose $F\colon [a,b]\to\mathbb{C}$.

(a) Define what it means for F to be absolutely continuous on $[a,b]$.

(b) Give an example of a function which is uniformly continous but not absolutely continuous. (Remember to justify your answer.)

(c) Prove that if there exists M such that $|F(x)-F(y)| \le M|x-y|$ for all $x, y\in[a,b]$, then F is absolutely continous. Is the converse true? (Again, remember to justify your answer.)