MATH 507a QUALIFYING EXAM September 24, 2014. One hour and 50 minutes, starting at 4pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let $E_1, E_2, \ldots$ be arbitrary events. Let $G := \limsup_n E_n = \{E_n \text{ i.o.}\}$.

(a) Show that $P(G) = 1$ if and only if $\sum_n \mathbb{P}(A \cap E_n) = \infty$ for all events A having $\mathbb{P}(A) > 0$.

(b) True or false: if $\mathbb{P}(G) = 0$ then $\sum_n \mathbb{P}(A \cap E_n) < \infty$ for all events A having $P(A) > 0$. If you think this is true, then provide a proof; if you think this is false, then provide a counter-example.

2. Let $\{X_n,\ n \geq 1\}$ and $\{Y_n,\ n \geq 1\}$ be two sequences of random variables such that $\lim_{n\to\infty} X_n = X$ in distribution for some random variable X with $\mathbb{P}(|X| < \infty) = 1$, and

$$\lim_{n\to\infty} \mathbb{P}(Y_n > c) = 1$$

for every $c > 0$.

Prove that

$$\lim_{n\to\infty} \mathbb{P}(X_n + Y_n > c) = 1$$

for every $c > 0$.

3. Let $a > 0$, let $X_n,\ n \geq 1$, be iid random variables that are uniform on $(0, a)$, and let $Y_n = \prod_{k=1}^n X_k$. Determine, with a proof, all values of a for which $\lim_{n\to\infty} Y_n = 0$ with probability one.