Preliminary Exam in Numerical Analysis Fall 2014

Instructions The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. Linear systems

(a) Let A be an $n \times n$ matrix, B be an $n \times m$ matrix, C be an $m \times n$ matrix, and let D be an $m \times m$ matrix with the matrices A and $D - CA^{-1}B$ nonsingular. Show that the partitioned $(n+m) \times (n+m)$ matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is nonsingular and find an explicit formula for its inverse.

(b) How many multiplications would be needed to compute M^{-1} if you do **not** take advantage of its special structure.

(c) How many multiplications would be needed to compute M^{-1} if you do take advantage of its special structure (i.e. using the formulas you derived in part (a)).

2. Least squares

Let $A \in R^{m\times n}$ and $b \in R^m$, with $m \leq n$ and rank $A = m$. Let $x_0 \in R^n$ and consider the constrained optimization problem given by

$$\min ||x - x_0||,$$

subject to $Ax = b$, where the norm in the above expression is the Euclidean norm on R^n. Show that this problem has a unique solution given by

$$x^* = A^T(AA^T)^{-1}b + (I_n - A^T(AA^T)^{-1}A)x_0,$$

where I_n is the $n \times n$ identity matrix.

3. Iterative Methods

Consider the following matrix

$$A = \begin{bmatrix} -2 & \frac{1}{2} \\ -\frac{1}{2} & -2 \end{bmatrix}.$$

(a) Find a range for the real parameter ω such that Richardson's method for the solution of the linear system $Ax = b$,

$$x^{k+1} = x^k - \omega\left(Ax^k - b\right)$$

converges for any initial guess x^0.

(b) Find an optimal value for ω and justify your answer.

4. Computation of Eigenvalues and Eigenvectors

Consider the following matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1+\varepsilon \end{bmatrix}.$$

(a) Find a non-singular matrix $T(\varepsilon)$ such that the matrix $T^{-1}(\varepsilon)AT(\varepsilon)$ is in Jordan canonical form.

(b) Find the limit of $\|T(\varepsilon)\|$ as ε tends toward zero.

(c) Explain what this implies with regard to using the LR or QR methods to compute the eigenvalues of a matrix whose Jordan canonical form contains a Jordan block of size larger than 1.