Geometry/Topology Qualifying Exam - Fall 2014

1. Show that if (X, x) is a pointed topological space whose universal cover exists and is compact, then the fundamental group $\pi_1(X, x)$ is a finite group.

2. Recall that if (X, x) and (Y, y) are pointed topological spaces, then the wedge sum (or 1-point union) $X \vee Y$ is the space obtained from the disjoint union of X and Y by identifying x and y. Show that T^2 (the 2-torus $S^1 \times S^1$) and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but are not homeomorphic.

3. Suppose S^n is the standard unit sphere in Euclidean space and that $f : S^n \to S^n$ is a continuous map.

   i) Show that if f has no fixed points, then f is homotopic to the antipodal map.

   ii) Show that if $n = 2m$, then there exists a point $x \in S^{2m}$ such that either $f(x) = x$ or $f(x) = -x$.

4. If M is a smooth manifold of dimension d, using basic properties of de Rham cohomology, describe the de Rham cohomology groups $H^*_{dR}(S^1 \times M)$ in terms of the groups $H^*_{dR}(M)$ (along the way, please explain, quickly and briefly, how to compute $H^*_{dR}(S^1)$).

5. Show that if $X \subset \mathbb{R}^3$ is a closed (i.e., compact and without boundary) submanifold that is homeomorphic to a sphere with $g > 1$ handles attached, then there is a non-empty open subset on which the Gaussian curvature K is negative.

6. Suppose M is a (non-empty) closed oriented manifold of dimension d. Show that if ω is a differential d-form, and X is a (smooth) vector field on X, then the differential form $\mathfrak{L}_X\omega$ necessarily vanishes at some point of M.

7. Let V be a 2-dimensional complex vector space, and write $\mathbb{CP}^1$ for the set of complex 1-dimensional subspaces of V. By explicit construction of an atlas, show that $\mathbb{CP}^1$ can be equipped with the structure of an oriented manifold.