MATH 505a QUALIFYING EXAM September 23, 2014. One hour and 50 minutes, starting at 2pm.

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let A and B be two events with $0 < \mathbb{P}(A) < 1$, $0 < \mathbb{P}(B) < 1$. Define the random variables $\xi = \xi(\omega)$ and $\eta = \eta(\omega)$ by

$$\xi(\omega) = \begin{cases} 5 & \text{if } \omega \in A; \\ -7 & \text{if } \omega \notin A; \end{cases} \qquad \eta(\omega) = \begin{cases} 2 & \text{if } \omega \in B; \\ 3 & \text{if } \omega \notin B. \end{cases}$$

True or false: the events A and B are independent if and only if the random variables ξ and η are uncorrelated? If you think this is true, then provide a proof. If you think this is false, then give a counter-example.

2. n people each roll one fair die. For each (unordered) pair of people that get the same number of spots, that number of spots is scored, with S for the total score achieved among the $\binom{n}{2}$ pairs of people. For example, if there are $n = 10$ people, and they roll 1,2,2,2,3,4,4,4,4,6 then $S = 2+2+2+4+4+4+4+4+4$ since there are three pairs of people matching 2 and six $= \binom{4}{2}$ pairs of people scoring 4.

(a) Simplify $\mathbb{E}S$.

(b) Simplify $\mathbb{E}S^2$.

[HINT: Consider S as the sum of $\binom{n}{2}$ random variables $S_{i,j}$, where $S_{i,j}$ is k if persons i and j both roll k, and zero otherwise.]

3. Let X be a standard normal random variable and, for $a > 0$, define the random variable Y_a by

$$Y_a = \begin{cases} X, & \text{if } |X| < a, \\ -X, & \text{if } |X| > a. \end{cases}$$

(a) Verify that Y_a is a standard normal random variable.

(b) Express $\rho(a) = \mathbb{E}(XY_a)$ in terms of the probability density function $\varphi = \varphi(x)$ of X.

(c) Is there a value of a for which $\rho(a) = 0$?

(d) Does the pair (X, Y_a) have a bivariate normal distribution? Explain your reasoning.