Algebra Exam September 2014

Show your work. Be as clear as possible. Do all problems.
Hand in solutions in numerical order.

1. Let G be a group of order 56 having at least 7 elements of order 7. Let S be a Sylow 2-subgroup of G.

   (a) Prove that S is normal in G and $S = C_G(S)$.

   (b) Describe the possible structures of G up to isomorphism. (Hint: How does an element of order 7 act on the elements of S?)

2. Show that a finite ring with no nonzero nilpotent elements is commutative.

3. If $R = M_n(\mathbb{Z})$, and A is an additive subgroup of R, show that as additive subgroups $[R : A]$ is finite if and only if $R \otimes_{\mathbb{Z}} \mathbb{Q} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

4. Let R be a commutative ring with 1, n a positive integer and $A_1, \ldots A_k \in M_n(R)$. Show that there is a noetherian subring S of R containing 1 with all the $A_i \in M_n(S)$.

5. Let $R = \mathbb{C}[x, y]$. Show that there exists a positive integer m such that $((x + y)(x^2 + y^4 - 2))^m$ is in the ideal $(x^3 + y^2, y^3 + xy)$.

6. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree $n \geq 5$. Let L be the splitting field of f and let $\alpha \in L$ be a zero of f. Given that $[L : Q] = n!$, prove that $\mathbb{Q}]\alpha^4] = \mathbb{Q}[\alpha]$.