DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2006

1. Let $f : R^n \to R^n$ be Lipshitz continuous and $x(t)$ a solution to the initial value problem $x' = f(x)$, $x(t_0) = x_0 \neq 0$. Show that if $x \cdot f(x) \geq \|x\|^3$ then the solution cannot extend to the interval $[t_0, \infty)$.

2. Consider the system $x' = h(t)Ax$, where $x(t)$ is an n-vector, A is a constant $n \times n$ matrix and $h(t)$ is strictly positive and continuous. Show that the trivial solution 0 is asymptotically stable if all eigenvalues of A lie in the left half plane and the integral $\int_0^\infty h(t)dt$ diverges. Also give an example that shows that the stability may not apply if the integral converges.

3. Show that at $(0,0)$ in the x, y plane, NO analytic Center Manifold, $y = f(x)$ exists for the following system:

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y + x^2. \end{aligned}$$

4. Let u be harmonic and bounded on $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n);\ x_n > 0\}$ and $u = 0$ on $\{(x_1, \ldots, x_n);\ x_n = 0\}$. Show that u is a constant.

5. Let $g(x)$ be a bounded and continuous function on $\mathbb{R}^n$, and

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) g(y)\, dy.$$

   (i) What is the PDE satisfied by $u(x,t)$?

   (ii) Show that $|u(x,t)| \leq \sup_y |g(y)|$.

   (iii) If in addition, $\int_{\mathbb{R}^n} |g(y)|\, dy < \infty$, show that $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in \mathbb{R}^n$.

6. Let u be a smooth solution of

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty) \qquad (1)$$
$$u = g,\ u_t = h \quad \text{on } \mathbb{R}^3 \times \{t = 0\} \qquad (2)$$

   where g and h are smooth and have compact support. Prove the existence of a positive constant C such that

$$|u(x,t)| \leq \frac{C}{t}$$

   for all $x \in \mathbb{R}^3$ and $t > 0$.

   Hint:

$$u(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} th(y) + g(y) + \nabla g(y) \cdot (y - x)\, dS(y)$$