Complex analysis, Graduate Exam Fall 2006

Answer all four questions. Partial credit will be given to partial solutions.

1. Let a be a real number with $a > e$. Show that the equation $e^z = az^n$ has n solutions inside the unit circle.

2. Find the Fourier transform $\hat{f}(\omega)$ of the function $f(x) = \frac{1}{1+x^2}$, $x \in \mathbb{R}$,

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{1+x^2}\, dx.$$

3. Fix $\tau \in \mathbb{C}$ with non-zero imaginary part. Let $f(z)$, $z \in \mathbb{C}$ be a non-constant meromorphic function such that $f(z) = f(z + m + n\tau)$ for all $m, n \in \mathbb{Z}$ (such functions are called doubly periodic). Show that f has infinitely many singularities.

4. Let $f(z)$ be a one to one conformal map from the unit disk to a square with center 0 such that $f(0) = 0$. Show that $f(iz) = if(z)$. Show also that if $f(z)$ is given by the power series $f(z) = \sum_{n=1}^{\infty} c_n z^n$ then $c_n = 0$ for all n such that $n - 1$ is not divisible by 4.