ALGEBRA QUALIFYING EXAM
September, 2006

1. (a) Find the number of Sylow p-subgroups of the symmetric group S_p. Here p is a prime.

(b) Use (a) to prove that

$$(p-1)! + 1 \equiv 0 \bmod p.$$

2. Let G be a finite solvable group and H a minimal (non-trivial) normal subgroup. Show H is isomorphic to a direct sum of cyclic groups of order p, for some prime p. (Hint: First show that the commutator subgroup H' of H is $\{e\}$.)

3. Let $m_1, m_2, \ldots, m_n$ be positive integers which are pairwise relatively prime (that is, $\gcd(m_i, m_j) = 1$ for all $i \neq j$).

(a) Show that $F := \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n}) = \mathbb{Q}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_n})$. (Hint: induction.)

(b) Show that $F = \mathbb{Q}(\sqrt{m_1} + \sqrt{m_2} + \cdots + \sqrt{m_n})$ is Galois over $\mathbb{Q}$. What is its Galois group?

4. Let k be a field, let R be a commutative k-algebra , and let $S = M_n(R)$ be all $n \times n$-matrices with entries in R. Choose $A_1, ..., A_m \in S$. Show that there exists a (left) Noetherian k-subalgebra S_0 of S which contains all of the matrices A_i. (Hint: Consider the subalgebra $R_0 \subset R$ generated by all entries of $A_1, ..., A_m \in S$.)

5. Let $R = \mathbb{C}[x, y, z]$, let $I = (x^2z^3 - y^2z + xyz - x^2y)$ be an ideal of R, and define $S = R/I$.

(a) Prove that the polynomial $x^2z^3 - y^2z + xyz - x^2y$ is irreducible in R. (Hint: consider it as an element of $\mathbb{C}[x, y][z]$.)

(b) Show that S is a Noetherian integral domain.

(c) Prove that in S the intersection of all maximal ideals is $\{0\}$.

6. Let k be a finite field and let R be a finite dimensional semi-simple k-algebra such that for all $r \in R$, there exists a positive integer $n = n(r) > 0$ such that $r^{n(r)}$ is in the center of R. Prove that R is commutative.

7. Let M be a finitely generated $\mathbb{Z}$-module with torsion submodule $T(M)$.

(a) Justify: $M/T(M)$ is a free $\mathbb{Z}$-module.

For parts (b) and (c), set $r(M) := \text{rank}(M/T(M))$.

(b) Show that $r(M) = \dim_{\mathbb{Q}}(M \otimes_{\mathbb{Z}} \mathbb{Q})$.

(c) Assume that

$$0 \to M \to N \to P \to 0$$

is an exact sequence of $\mathbb{Z}$-modules. Show that

$$r(N) = r(M) + r(P).$$