DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2010

The exam has six problems.

1. Consider the product space $\mathbb{R}^n \times \mathbb{R}$ and let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, $P(x,t) = t$. For two t values, $t_1 < t_2$ define the "copies" of $\mathbb{R}^n$, $X = P^{-1}(t_1)$ and $Y = P^{-1}(t_2)$ and the mapping

$$T : X \to Y \quad \text{where} \quad y = T(x) = \phi(t_2, t_1, x)$$

where $\phi(t_2, t_1, x)$ is the solution $\phi(t, t_1, x)$ of $y' = f(t, y), \quad y(t_1) = x$, evaluated at t_2. Assume that $f \in C^1$ and the divergence,

$$\text{div } f \doteq \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_2} + \cdots + \frac{\partial f_n}{\partial y_n} = 0.$$

Show (1) T is a diffeomorphism and (2) T is (Borel) measure preserving. **Note:** Let $(X, \mathcal{B}, \mu)$ be a measurable space. Then $h : X \to X$ is measure preserving if $\mu(h^{-1}(U)) = \mu(U)$ for each $U \in \mathcal{B}$. If h is invertible then h is measure preserving if h^{-1} is measure preserving, i.e. if $\mu(h(U)) = \mu(U)$ for each $U \in \mathcal{B}$.

2. Let

$$\frac{dx}{dt} = f(x), \qquad f \in C^1(\mathbb{R}^3, \mathbb{R}^3)$$

and suppose $x = \phi(t)$ is the solution that satisfies $\|\phi(0)\| = 2$ and $\|\phi(t)\| \le 1$ for $t \ge t_0 > 0$.

   (a) Prove there is a subset of the unit ball that is invariant under the flow generated by the ODE.

   (b) Prove there is a solution $\psi(t)$ that satisfies $\|\psi(0) - \psi(t_n)\| \to 0$ for some sequence $t_n \to \infty$.

3. Consider the planar system

$$\begin{aligned} r' &= r(1-r) \\ \theta' &= r \sin^2(\frac{\theta}{2}). \end{aligned}$$

(a) Find the α and ω limit sets of all points in the plane.

(b) Is the stationary point $(r, \theta) = (1, 0)$ asymptotically stable?

4. Consider the wave equation in $\mathbb{R}^3$

$$u_{tt} - \Delta u = 0 \quad \text{for } x \in \mathbb{R}^3,\ t > 0, \tag{1}$$
$$u(x,0) = 0, \tag{2}$$
$$u_t(x,0) = g(x), \tag{3}$$

where $g \in C_0^\infty(\mathbb{R}^3)$. Prove that there exists a constant C depending only on the given data such that

$$\sup_{x\in\mathbb{R}^3} |u(x,t)| \le \frac{C}{t}, \quad t > 0.$$

5. Use the method of characteristics to solve the following partial differential equation:

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 3u, \quad u(x,0) = u_0(x). \tag{4}$$

6. Let $u(x)$ be harmonic for $x \in \mathbb{R}^3$. Suppose that $\nabla u \in L^2(\mathbb{R}^3)$. Prove that u is a constant.