DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2009

1. Consider the initial value problem for $x, y \in \mathbb{R}$,
$$\frac{dy}{dx} = 2(2+x)(1+y^2), \qquad y(0) = 0.$$
Determine the value of x, if one exists, at which the solution attains its minimum value. **Hint:** Solve the equation and study the maximal interval of existence

2. Consider the systems, for $t \in \mathbb{R}^+$,
$$\frac{dy}{dt} = A(t)y, \qquad \text{and} \qquad (1)$$
$$\frac{dy}{dt} = [A(t) + B(t)]y. \qquad (2)$$
Assume
$$\int_0^\infty \|B(t)\| dt < \infty, \qquad \text{and} \qquad \liminf_{t\to\infty} \int_0^t trA(s)ds > -\infty.$$
If all the solutions of (1) are bounded on $\mathbb{R}^+$ show all the solutions of (2) are bounded on $\mathbb{R}^+$.

3. Let $I = (-\infty, \infty)$ and $\Phi(t) = \begin{pmatrix} t^2 & 1 & 0 \\ t+1 & t^2+1 & 2 \\ t+4 & 0 & t+2 \end{pmatrix}$. Prove that Φ cannot be a fundamental matrix for a linear homogeneous system $x' = A(t)x$ defined on all of I. What happens to the above conclusion when you are allowed to choose the interval $I = (a, b),\ a < b$?

4. Prove that if $u \in W^{1,p}(0,1)$ for some $1 < p < \infty$, then
$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'(t)|^p \, dt \right)^{1/p}$$
for a.e. $x, y \in [0, 1]$.

5. Solve the Cauchy problem
$$u_x u_y = u, \qquad \text{with } \ u(0, y) = y^2.$$

6. Let $u(x,t)$ be a solution of the one-dimensional heat equation $u_t = u_{xx}$ with initial data
$$u(x,0) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0. \end{cases}$$
Find
$$a = \lim_{t\to\infty} u(x,t)$$
$$b = \lim_{x\to\infty} u(x,t)$$
$$c = \lim_{x\to-\infty} u(x,t)$$