DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2011

1. Brouwer's Fixed Point Theorem tells us that if
$$D = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$$
and $f : D \to D$ is continuous then f has a fixed point in D. Obtain this same conclusion for $f \in C^1$ by finding an equilibrium for the vector field $g(x) = f(x) - x$.

2. Consider the autonomous system $x'(t) = f(x(t))$, where $x = (x_1, \dots, x_n)$ and $f = (f_1, \dots, f_n)$ is a smooth vectorfield such that $\sum_k x_k f_k(x) < 0$ for all $x \neq 0$. Show that $x(t) \to 0$ as $t \to \infty$ for all solutions of the system (and all initial data $x(0)$).

3. Let
$$f(x,y) = \begin{pmatrix} ax - bxy - ex^2 \\ -cy + dxy - fy^2 \end{pmatrix},$$
where $a, b, c, d, e, f > 0$. Show that the system $(x', y') = f(x, y)$ has no closed orbit in the first quadrant. (Hint: Show that the divergence of $\frac{1}{xy} f(x, y)$ is non-zero.)

4. Let Ω be an open subset of $\mathbb{R}^n$. Suppose $u \in C^2(\bar{\Omega})$ is a solution of the equation $\Delta u = u^3$ with the property that $|\nabla u(x)| \leq 1$ for each $x \in \partial\Omega$. Prove that $|\nabla u(x)| \leq 1$ for all $x \in \Omega$.

5. Let Ω be an open bounded subset of $\mathbb{R}^n$, and assume $u(x,t) \geq 0$ is a function in $C^2(\bar{\Omega} \times [0, \infty)$ which solves the heat equation with heat loss due to radiation
$$u_t - \Delta u = -u^4$$
with the boundary condition $u = 0$ on $\partial\Omega$. Prove that we can find a constant C such that
$$E(1) = \int_\Omega u^2(x, 1)\, dx \leq C$$
regardless of the initial value $u(x, 0)$.

6. Solve the Cauchy problem
$$x\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -xy$$
$$u = 5 \quad \text{on} \quad xy = 1, x, y > 0$$