Spring 2010 Math 541b Exam

1. Consider the multinomial model with cell probabilities $p = (p_1, p_2, \cdots, p_m) \in \Omega$, in which $\sum_{i=1}^m p_i = 1$, $p_i > 0$, for $i = 1, \cdots, m$. And the count data generated from a multinominal is denoted by $x = (x_1, x_2, \cdots, x_m)$, where $\sum_{i=1}^m x_i = n$. Under the null hypothesis H_0, the vector of cell probabilities p is specified by $p = p(\theta) \in \omega_0$, where $\theta \in \Theta$. One example is the parameter specification in the Pearson's chi-square test of independence. In the alternative model H_A, the parameter vector $p \in \Omega - \omega_0$.

   (a) Denote $\hat{p}_i = x_i/n$, and the maximum likelihood estimate of θ under H_0 by $\hat{\theta}$. Show that the log-likelihood ratio test statistic is

   $$-2\log\Lambda = 2\sum_{i=1}^{m} O_i \log(\frac{O_i}{E_i}),$$

   where $O_i = n\hat{p}_i$ and $E_i = np(\hat{\theta})$.

   (b) Assume that $\dim\omega_0 = k$, what is the asymptotic distribution of the likelihood ratio test statistic as $n \longrightarrow +\infty$.

   (c) What is the Pearson test statistic in this general scenario?

   (d) Show that the likelihood ratio test statistic and the Pearson test statistic are approximately equivalent. (Hint: You may use a Taylor expansion.)

2. Consider the Bernoulli-Laplace model in which there are two urns, each containing M balls, and of these $2M$ total balls M are black and M are white. Suppose at each time step, one ball is chosen from each urn at random and they are interchanged. Let $X_n \in \{0, \ldots, M\}$ be the number of black balls in the first urn just after the nth step.

   (a) Find the transition probabilities of the Markov Chain X_n.

   (b) Find the stationary distribution of X_n. Prove stationarity.