Fall 2010 Math 541b Exam

1. Let $p(x)$ and $q(x)$ be two distinct density functions that are positive on $\mathbb{R}$.

   (a) We define the Kullack Leibler divergence as

   $$D(p||q) = E_p \log \frac{p(x)}{q(x)},$$

   where E_p means taking the expectation under density $p(x)$. Prove that $D(p||q)$ is strictly positive.

   (b) Let $X_1, \ldots, X_n$ be i.i.d. with density

   $$p(x; \theta) = \begin{cases} p(x) & \theta = 0 \\ q(x) & \theta = 1. \end{cases}$$

   For some fixed $\delta > 0$, consider the test $H_0 : \theta = 0$ versus $H_1 : \theta = 1$ which has rejection region A_n given by

   $$A_n = \{(X_1, \ldots, X_n) : e^{n(D(p||q)-\delta)} \leq \prod_{i=1}^{n} \frac{p(X_i)}{q(X_i)} \leq e^{n(D(p||q)+\delta)}\}^c.$$

   Prove that the sequence of Type I errors $P(A_n)$ tend to zero as $n \to \infty$, where $P(\cdot)$ is the probability under $p(x)$.

   (c) With $Q(\cdot)$ the probability under $q(x)$, prove that the Type II errors $Q(A_n^c)$ satisfy

   $$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log Q(A_n^c) = -D(p||q).$$

2. Suppose that X follows a Poisson distribution $\mathcal{P}(\theta)$, $\theta > 0$. Let θ have a Gamma $\Gamma(p, \lambda)$ distribution.

   (a) Show that the posterior distribution is $\Gamma(p + x, 1 + \lambda)$

   (b) Show that if we take the loss function as $l(\theta, a) = (a - \theta)^2$, then the Bayes estimate of θ is $(p + x)/(1 + \lambda)$.

   (c) Find the Bayes estimate if the loss function is $l(\theta, a) = (a - \theta)^2/\theta$