Fall 2009 Math 541b Exam

1. With $\mathbf{X}_1, \ldots, \mathbf{X}_n$ i.i.d. with density $p(\mathbf{x}; \theta), \theta \in \mathbb{R}^d$, consider the usual likelihood ratio test for $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ based on

$$\Lambda_n = \frac{\sup_{\theta \in \Theta_0} \prod_{i=1}^n p(\mathbf{x}_i; \theta)}{\sup_{\theta \in \Theta_0 \cup \Theta_1} \prod_{i=1}^n p(\mathbf{x}_i; \theta)}.$$

Let

$$\lambda_n = -2 \log \Lambda_n.$$

(a) Under sufficient regularity there exists Y such that,

$$\lambda_n \to_d Y \quad \text{as } n \to \infty.$$

State the distribution of Y.

(b) In parts b) and c) assume $\theta = (\theta_1, \theta_2)$, and let $p(\mathbf{x}; \theta)$ be the density of the bivariate normal $\mathbf{X}$ whose components have unknown mean (θ_1, θ_2), known variances σ_1^2, σ_2^2 and known correlation coefficient $\rho \in (-1, 1)$. For testing with $\Theta_0 = \{\theta_1 = 0, \theta_2 = 0\}$ and $\Theta_1 \cup \Theta_2 = \mathbb{R}^2$, verify that the distribution of λ is as specified in part a).

(c) For testing with $\Theta_0 = \{\theta_1 = 0, \theta_2 = 0\}$ and $\Theta_1 \cup \Theta_2 = \{\theta_1 > 0, \theta_2 \in \mathbb{R}\}$, determine the distribution of λ. Compare this distribution to the one in part b), and explain.

(d) State whether we can generalize the distributional result in c) asymptotically under the regularity assumed for the convergence in a), and justify your answer.

2. Let $X_1, \ldots, X_n$ be i.i.d. from the exponential distribution $\mathcal{E}(a, b)$ with density

$$b^{-1} \exp\{-(x - a)/b\} \cdot \mathbf{1}\{x \geq a\}, \quad -\infty < a < \infty, \quad b > 0,$$

and let $X_{(1)} = \min\{X_1, \ldots, X_n\}$.

(a) Determine the uniformly most powerful (UMP) test for testing $H_0 : a = a_0$ vs. $H_1 : a \neq a_0$ when b is assumed known.

(b) Show that the most powerful level-α test of H_0 vs. $H_1' : a = a_1$, for some given $a_1 < a_0$, has power equal to

$$\beta(a_1) = 1 - (1 - \alpha)e^{-n(a_0 - a_1)/b}. \tag{1}$$

(c) Show that, for the problem in part (a) but with b unknown, any level-α test which rejects when

$$\frac{X_{(1)} - a_0}{\sum(X_i - X_{(1)})} \notin (C_1, C_2) \tag{2}$$

is most powerful at all alternatives (a_1, b) with $a_1 < a_0$ (independent of the particular choice of C_1, C_2).