Spring 2010 Math 541a Exam

1. Let $X_1, \cdots, X_n$ be i.i.d. $\Gamma(p, 1/\lambda)$ with density $g_\theta(x) = \frac{1}{\Gamma(p)} \lambda^p \, x^{p-1} e^{-\lambda x}$, $x > 0$, $\theta = (p, \lambda)$, $p > 0$, $\lambda > 0$.

   (a) Find a moment estimate of the parameter.

   (b) Show that the moment estimates, $\tilde{\theta}$ are asymptotically bi-variate normal and give their asymptotic mean and variance covariance matrix.

   (c) Compute the asymptotic variance covariance matrix of the maximum likelihood estimates. You may leave your answer in terms of Γ function derivatives.

2. Recall that the t-distribution with $k > 0$ degrees of freedom, location parameter ℓ, and scale parameter $s > 0$ has density

$$\frac{\Gamma\left((k+1)/2\right)}{\Gamma(k/2)\sqrt{k\pi s^2}} \left\{ 1 + k^{-1} \left(\frac{x - \ell}{s} \right)^2 \right\}^{-(k+1)/2} .$$

   Show that the t-distribution can be written as a mixture of Gaussian distributions by letting $X \sim N(\mu, \sigma^2)$, $\tau = 1/\sigma^2 \sim \Gamma(\alpha, \beta)$, and integrating the joint density $f(x, \tau|\mu)$ to get the marginal density $f(x|\mu)$. What are the parameters of the resulting t-distribution, as functions of μ, α, β?