Fall 2010 Math 541a Exam

1. Let $\mu > 0$ be an unknown parameter, and suppose that X_1 and X_2 are independent random variables, each having the exponential distribution with density function

$$p(x;\mu) = \frac{1}{\mu} \exp\left(-\frac{x}{\mu}\right), \quad x > 0.$$

It is clear that since the distribution $p(x;\mu)$ has mean μ that an unbiased estimator of μ is given by $\overline{X} = (X_1 + X_2)/2$.

   (a) Calculate the variance of the estimator $\overline{X}$ of μ.

   (b) Now consider the estimator of μ given by $T(X_1, X_2) = \sqrt{X_1 X_2}$. Calculate the bias of $T(X_1, X_2)$.

   (c) Show that the mean square error of $T(X_1, X_2)$ as an estimator of μ, that is

$$\text{MSE}(T) = E(T(X_1, X_2) - \mu)^2,$$

   is strictly smaller than the mean squared error of the estimator $\overline{X}$ of μ.

2. A random variable X has the Weibull(α, β) distribution if

$$P_{\alpha,\beta}(X > x) = \exp\{-(x/\alpha)^\beta\} \quad \text{for } x \geq 0. \tag{1}$$

Suppose that $X_1, \ldots, X_n$ are i.i.d. Weibull(α_0, β_0), where (α_0, β_0) is in the interior of a compact parameter space $\Theta \subseteq \mathbb{R}^+ \times \mathbb{R}^+$.

   (a) Show that the MLE of α is given by

$$\widehat{\alpha} = \left(n^{-1} \sum_{i=1}^{n} X_i^{\widehat{\beta}}\right)^{1/\widehat{\beta}}, \tag{2}$$

   where $\widehat{\beta}$ is the MLE of β.

   (b) As an alternative to maximum likelihood, let $\widetilde{\beta}$ be any estimator of β and consider the estimator of α given by

$$\widetilde{\alpha} = \left(n^{-1} \sum_{i=1}^{n} X_i^{\widetilde{\beta}}\right)^{1/\widetilde{\beta}}. \tag{3}$$

Show that $\widetilde{\alpha}$ is a "pseudo-MLE" in the sense that it maximizes $\ell(\alpha, \widetilde{\beta})$, where $\ell(\alpha, \beta)$ is the log-likelihood function.

(c) Prove that $EX_1^{\beta_0} = \alpha_0^{\beta_0}$.

(d) Prove that if $\widetilde{\beta}$ is any consistent estimator, than $\widetilde{\alpha}$ given by (3) is consistent for α. *Hint:* Letting $Y_n(\beta) = n^{-1} \sum_i X_i^{\beta}$, argue that it suffices to prove that $Y_n(\widetilde{\beta}) \to \alpha^{\beta_0}$ in probability. Then, using the convexity of $Y_n(\beta)$, show that

$$\left| Y_n(\widetilde{\beta}) - Y_n(\beta_0) \right| \to 0 \quad \text{in probability,}$$

and complete the argument by applying part (2c) and the law of large numbers.