**Spring 2011 Math 541a Exam**

1. Let $X_1, \ldots, X_n$ be i.i.d. with distribution $\mathcal{N}(\mu, \sigma^2)$ and $n \geq 2$.

   (a) Find UMVU estimates $\widehat{\mu}$ and $\widehat{\sigma^2}$ of $\mu$ and $\sigma^2$, respectively, and prove that they are such.

   (b) Derive the marginal distributions of $\widehat{\mu}$ and $\widehat{\sigma^2}$, and prove that these estimators are independent.

2. For $\theta \in \mathbb{R}$ let $X_1, X_2, \cdots, X_n$ be independent continuous random variables, each having density function

$$p(x; \theta) = \exp(-(x - \theta))I\{x > \theta\},$$

   where $I(x) = 1$ if $x > 0$ and $I(x) = 0$ otherwise. Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the corresponding order statistics.

   (a) Find the joint density function of $(X_{(1)}, X_{(2)})$, and the marginal densities of $X_{(1)}$ and $X_{(2)}$.

   (b) Show that

$$T = X_{(1)} - (n-1)(X_{(2)} - X_{(1)})/n$$

   is an unbiased estimator of $\theta$.

   (c) Find the maximum likelihood estimate of $\theta$.