Geometry/Topology Qualifying Exam

Spring 2011

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Let $S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$ be the 3-dimensional sphere, oriented as the boundary of the unit ball B^4 in $\mathbb{R}^4$ with the standard orientation. Compute $\int_{S^3} \omega$, where

   $$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.$$

   (You may leave your answer in terms of volumes $vol(S^n)$ and $vol(B^n)$.)

2. (10 pts) Let $M = \{(x, y) \mid x, y \in \mathbb{R}^3, \|x\| = 1, \|y\| = 1, \langle x, y\rangle = 0\}$, where $\langle x, y\rangle$ is the standard inner product on $\mathbb{R}^3$. Show that M is a smooth compact embedded submanifold of $\mathbb{R}^6$ and explain how M can be identified with the unit tangent bundle of S^2.

3. (20 pts) Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.
   (a) (5 pts) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$.
   (b) (5 pts) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geq 1$.
   (c) (5 pts) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$, $k \geq 0$.
   (d) (5 pts) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

4. (10 pts) Given a continuous map $f : X \to Y$ between topological spaces, define

   $$C_f = \Big((X \times [0,1]) \coprod Y\Big) / \sim,$$

   where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Then prove that there is a long exact sequence

   $$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \widetilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$

   where f_* is the map on homology induced from f and $\widetilde{H}_i$ denotes the ith reduced homology group.

5. (10 pts) Prove that the fundamental group of a connected Lie group G is abelian. (A *Lie group* G is a smooth manifold which is also a group, and whose group operations multiplication and inverse are smooth maps.) [Hint: One possible way of proving this is to find an explicit homotopy between fg and gf, where f and g are loops in G.]

6. (10 pts) Let $M \subset \mathbb{R}^3$ be an embedded compact oriented surface (without boundary) of genus $g \geq 1$. Show that the Gaussian curvature κ of M must vanish somewhere on M.