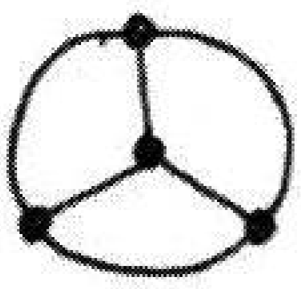
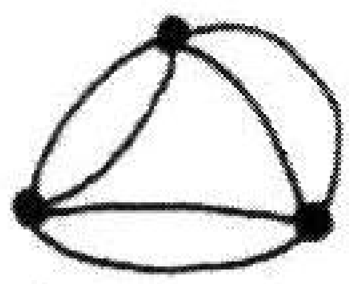
Geometry and Topology Graduate Exam
Fall 2010

Problem 1. Compute the fundamental groups of the following two graphs:

$X_1 =$  $X_2 =$ 

Problem 2. Let P_1, P_2, P_3 be three distinct points in the sphere S^2, and let X be the topological space obtained from S^2 by gluing these three points together. Compute all homology groups $H_n(X;\mathbb{Z})$.

Problem 3. Define the Gaussian (or scalar) curvature $\kappa(p)$ of an immersed surface Σ in $\mathbb{R}^3$ at the point p. Does there exist a compact immersed surface Σ without boundary in $\mathbb{R}^3$ which has $\kappa(p) = -1$ for all $p \in \Sigma$?

Problem 4. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the *orthogonal group* $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = \mathrm{id}\}$ is a smooth manifold. What is its dimension?

Problem 5. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be a differential form such that

$$d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where x_1, x_2, ..., x_n are the standard coordinates of $\mathbb{R}^n$. Show that, for every $p \in \mathbb{R}$, the differential form

$$\alpha = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^p}\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Possible hint: S^{n-1}.

Problem 6. Consider the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates x_1, y_1, ..., x_n, y_n. If f is a smooth function on $\mathbb{R}^{2n}$, find the vector field X such that $i_X\omega = df$, where i_X denotes the interior product. Then compute the Lie derivative $\mathcal{L}_X\omega$.

Problem 7. Let X be a topological space such that the homology group $H_p(X;\mathbb{Z})$ is finite and such that the cohomology group $H^{p+1}(X;\mathbb{Q})$ is equal to 0. Let $u \in C^{p+1}(X;\mathbb{Z}) = \mathrm{Hom}\big(C_{p+1}(X;\mathbb{Z}),\mathbb{Z}\big)$ be a cochain with $du = 0$.

a. Show that, for every $\alpha \in C_p(X,\mathbb{Z})$ with $\partial\alpha = 0$, there exists $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

b. Show that there exists a homomorphism

$$L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$L_u([\alpha]) = \frac{1}{k}u(\beta)$$

for every $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$. Namely, show that $L_u([\alpha])$ is independent of k, β and of the representative α of $[\alpha] \in H_p(X;\mathbb{Z})$.