Geometry and Topology Graduate Exam
Fall 2009

Problem 1. Let $f : M \to N$ be a map between two compact oriented manifolds of the same dimension. Suppose that the subgroup $f^*(\pi_1(M))$ has finite index in $\pi_1(N)$.

a. Show that the index $[\pi_1(N) : f^*(\pi_1(M))]$ divides the degree of f.
b. Give an example where $[\pi_1(N) : f^*(\pi_1(M))]$ is different from the degree of f.

Problem 2. Is there a differentiable map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends the vector field $\frac{\partial}{\partial x}$ to the vector field $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and sends the vector field $\frac{\partial}{\partial y}$ to the vector field $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$?

Problem 3. Let $f : S^n \to S^n$ be a degree 5 map from the sphere S^n to itself.

a. Show that there exists $x_1 \in S^n$ such that $f(x_1) = -x_1$.
b. Show that there exists $x_2 \in S^n$ such that $f(x_2) = x_2$.

Problem 4. Let M be a compact submanifold of $\mathbb{R}^n$, of dimension at most $n - 3$, and let $f : B^2 \to \mathbb{R}^n$ be a differentiable map from the 2–dimensional ball (or disk) B^2 to $\mathbb{R}^n$. Let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation along the vector $v \in \mathbb{R}^n$.

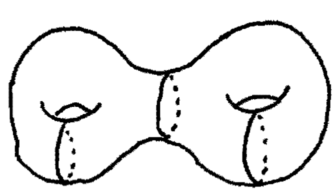
a. Show that there exists arbitrarily small vectors $v \in \mathbb{R}^n$ such that the image of $T_v \circ f$ is disjoint from M.
b. Conclude that the complement $\mathbb{R}^n - M$ is simply connected.

Problem 5. Let ω be a closed form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Show that, for any two differentiable maps $f, g : S^n \to \mathbb{R}^{n+1} - \{0\}$, the ratio

$$\frac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)}$$

is a rational number when the denominator is not 0.

Problem 6. Let S be the standard surface of genus 2 in $\mathbb{R}^3$ as in the picture below, and let W be the closure of the bounded component of $\mathbb{R}^3 - S$. Compute the relative homology groups $H_n(W, S)$.



Problem 7. Let M be a compact connected submanifold of an oriented manifold N, with $\dim M = \dim N - 1$. Show that M is orientable if and only if it admits arbitrarily small connected neighborhoods U such that $U - M$ is disconnected. Namely, if and only if, for every open subset $V \subset N$ containing M, there is a connected open subset $U \subset V$ such that $U - M$ is not connected.