REAL ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let f be a bounded function on $\mathbb{R}^n$ and for $\epsilon > 0$ let $M_\epsilon(x) = \sup_{y:|y-x|<\epsilon} f(y)$.
  (a) Show that $M(x) = \lim_{\epsilon \to 0} M_\epsilon(x)$ exists for all x.
  (b) Show that M is upper semicontinuous, that is, $\limsup_{y \to x} M(y) \leq M(x)$.

(2) Let m be Lebesgue measure on $[0, 1]$, suppose $f \in L^1(m)$ and let $F(x) = \int_0^x f(t)\, dt$. Suppose φ is a Lipshitz function, that is, for some M, $|\varphi(x) - \varphi(y)| \leq M|x - y|$ for all x, y. Show that there exists $g \in L^1(m)$ such that $\varphi(F(x)) = \int_0^x g(t)\, dt$.

(3) Let χ_E denote the indicator function of a set E. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and define

$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y - x)\, dy.$$

Show that f is continuous.

(4) Let m be Lebesgue measure on $\mathbb{R}$ and let $f_n, f \in L^1(m)$. Suppose there is a constant C such that $||f_n - f||_1 \leq \frac{C}{n^2}$ for all $n \geq 1$. Show that $f_n \to f$ a.e. HINT: Consider the sets

$$\{x : |f_n(x) - f(x)| > \epsilon \text{ for some } n \geq N\}.$$