REAL ANALYSIS GRADUATE EXAM
Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $f \geq 0$ and suppose $f \in L^1([0,\infty))$. Find

$$\lim_n \frac{1}{n} \int_0^n x f(x)\, dx.$$

(2) Suppose $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show that f^α is absolutely continuous.

(3)(a) Let $\{\mu_k\}$ be a sequence of finite signed measures. Find a finite positive measure μ such that $\mu_k \ll \mu$ for all k.

(b) Construct an increasing function whose set of discontinuities is $\mathbb{Q}$. (Prove it is a valid example.)

(4) Let m be Lebesgue measure on $\mathbb{R}$. For $f \in L^1_{loc}$ and $x \in \mathbb{R}^n$, define the function $A_r f$ by

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy,$$

which is the average value of f on the ball $B(x,r)$ of radius r centered at x, and define the function Hf by $Hf(x) = \sup_{r>0} A_r|f|(x), x \in \mathbb{R}^d$.

(a) Show that for $f \in L^1(\mathbb{R}^n), f \neq 0$, there exist $C, C', R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for all $|x| > R$ and

$$m\Big(\{x : Hf(x) > \alpha\}\Big) \geq \frac{C'}{\alpha} \quad \text{for all sufficiently small } \alpha.$$

(b) Define the function H^*f by

$$H^*f(x) = \sup\left\{\frac{1}{m(B)} \int_B |f(y)|\, dy : B \text{ is a ball containing } x\right\}.$$

Show that $Hf \leq H^*f \leq 2^n Hf$. (Note that unlike Hf, in the definition of H^*f the ball B need not be centered at x.)