REAL ANALYSIS GRADUATE EXAM
Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, and suppose each set in $\mathcal{A}$ has strictly positive measure. Show that $\mathcal{A}$ is at most countable.

(2)(a) Let m denote Lebesgue measure on $\mathbb{R}$ and let f be an integrable function. Show that for $a > 0$,

$$\int f(ax)\ m(dx) = \frac{1}{a} \int f(x)\ m(dx).$$

HINT: Consider a restricted class of functions f first.

(b) Let F be a measurable function on $\mathbb{R}$ satisfying $|F(x)| \leq C|x|$ for all x, and suppose F is differentiable at 0. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}\ m(dx) = \pi F'(0).$$

HINT: Use (a).

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let f be a measurable function with $|f| < 1$. Prove that

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n)\ d\mu$$

exists (it may be ∞.) HINT: First consider $f \geq 0$.

(4) Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a, b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$ is finite for all $x \in [a, b]$. Show that

$$F'(x) = \sum_{j=1}^{\infty} F_j'(x) \quad \text{for } m\text{-a.e. } x \in [a, b].$$

HINT: Consider the corresponding measures μ_F and μ_{F_j}.