REAL ANALYSIS GRADUATE EXAM
Spring 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $A \subset \mathbb{R}$ and suppose that for each $\epsilon > 0$ there are Lebesgue-measurable sets E, F with $E \subset A \subset F$ and $m(F \backslash E) < \epsilon$. Show that A is Lebesgue measurable.

(2) Let $f > 0$ be a Lebesgue-integrable function on $[0,1]$. Show that

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_{[0,1]} (f^\epsilon - 1)\ dm = \int_{[0,1]} \log f\ dm.$$

Here m denotes Lebesgue measure. HINT: Decompose f (or $\log f$) into two parts.

(3) Suppose $f \in L^1(\mathbb{R})$ is absolutely continuous, and

$$\lim_{h \searrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|\ dx = 0.$$

Show that $f = 0$ a.e.

(4)(a) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose $F_1, \ldots, F_7$ are 7 measurable sets with $\mu(F_j) \geq 1/2$ for all j. Show that there exist indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \phi$.

(b) Let m denote Lebesgue measure on $[0,1]$, and let $f_n \in L^1(m)$ be nonnegative and measurable with

$$\int_{[0,1/n]} f_n\ dm \geq 1/2$$

for all $n \geq 1$. Show that $\int_{[0,1]} [\sup_n f_n(x)]\ m(dx) = \infty$. HINT: Part (b) does not necessarily use part (a).