COMPLEX ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Notation: $\Re z$ denotes the real part of the complex number z, and $\Im z$ its imaginary part.

(1) Map the region $\Omega = \{\Im z > 0\} \setminus \{iy : 0 < y \leq 1\}$ conformally to the unit disk $D = \{|z| < 1\}$.

(2) How many zeroes of $p(z) = z^4 + z^3 + 4z^2 + 2z + 7$ lie in the right half plane $\{\Re z > 0\}$?

(3) Let f be analytic in the unit disk $D = \{|z| < 1\}$ and continuous on its closure $\bar{D}$. Show that if f is real valued on the boundary $\partial D = \{|z| = 1\}$ then f must be a constant.

(4) By consideration of $\int e^{z+\frac{1}{z}}\,dz$, or otherwise, show that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos\theta\,d\theta = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \cdots .$$