COMPLEX ANALYSIS GRADUATE EXAM
Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta + 2\sin\theta}.$$

(2) Suppose the series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Show that for $r < R$,

$$\int_{\{|z|=r\}} |f(z)|^2 \, dz = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

(3) Let $f(z)$ be analytic on $\mathbb{C}$ and suppose that the line $\Gamma = \{t + it : t \in \mathbb{R}\}$ is mapped to itself, that is, $f(z) \in \Gamma$ for all $z \in \Gamma$. If $f(\sqrt{2}) = 3$, then what is $f(\sqrt{2}i)$?

(4) Let $\Omega \subset \mathbb{C}$, with $\Omega \neq \mathbb{C}$, be simply connected, and let $f : \Omega \to \Omega$ be a conformal bijection. If f has two distinct fixed points z_1, z_2 (that is, $f(z_1) = z_1, f(z_2) = z_2$), show that f is the identity map.