COMPLEX ANALYSIS GRADUATE EXAM
Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Show that
$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\, dx = \frac{\pi(1-e^{-1})}{2}.$$

2. Suppose that f is holomorphic in a neighborhood of 0 and that

$(*)$ $$\sum_{n=0}^\infty f^{(n)}(z)$$

is absolutely convergent at $z=0$. Show that f is an entire function, and that $(*)$ is convergent for all $z \in \mathbb{C}$.

3. Let f be a non-negative real valued harmonic function in the disc $D = \{z \in \mathbb{C} : |z| < R\}$.

    (i) Prove that
$$\frac{R-|z|}{R+|z|}f(0) \le f(z) \le \frac{R+|z|}{R-|z|}f(0) \quad \text{whenever } |z| < R.$$

[Hint: use the Poisson formula.]

    (ii) Prove that
$$\frac{1}{3}f(0) \le f(z) \le 3f(0) \quad \text{whenever } |z| \le R/2.$$

    (iii) Let K be a compact subset of the open disc D. Show that there is a constant M depending only on K and R such that
$$f(z_1) \le M f(z_2) \quad \text{for all } z_1, z_2 \in K.$$

4. Liouville's theorem states that a bounded entire function f is constant.

    i) Give a proof of Liouville's theorem. You may use standard results about holomorphic functions such as Cauchy's theorem and power series representation, but any result you use should be clearly stated

    (ii) Suppose instead that f is entire and that $|f(z)| \le K(1+|z|^n)$ for some $K < \infty$ and positive integer n. Show that f is a polynomial of degree at most n.