COMPLEX ANALYSIS GRADUATE EXAM
Fall 2009

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Map the region $\Omega = \{z = x + iy : |z| < 1, y > 1/\sqrt{2}\}$ conformally to the unit disk $D = \{z : |z| < 1\}$.

(2) Let $f(z)$ be an entire function so that $f(z)$ does not assume any value in $[0, \infty)$. Show that f is a constant.

(3) Evaluate the integral $\displaystyle\int_0^\infty \frac{dx}{(1 + x^2)\sqrt{x}}$.

(4) Show that the infinite product $\displaystyle\prod_{k=0}^\infty (1 + z^{2^k})$ converges for $|z| < 1$ and equals $\dfrac{1}{1 - z}$ (for $|z| < 1$).

(5) Show that if $f : D \to D$ (D the unit disk) has two distinct fixed points, then $f(z) = z$.