COMPLEX ANALYSIS GRADUATE EXAM

Spring 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\, dx.$$

2. (i) Suppose that $u_1, u_2, \ldots, u_n$ and $u_1^2 + \cdots + u_n^2$ are harmonic functions on a connected open set D. Show that each function u_r $(1 \le r \le n)$ is constant.

(ii) A function $f : D \to \mathbb{C}$ with $f(x+iy) = u(x,y) + iv(x,y)$ is said to be complex harmonic if the real valued functions u and v are harmonic. Show that if $f(x+iy)$ and $(x+y)f(x+iy)$ are both complex harmonic then f is analytic.

3. Let $f : D \to D$ be an analytic function on a bounded domain D with $0 \in D$. Assume $f(0) = 0$ and $|f'(0)| < 1$. Let $F_n(z) = f \circ \cdots \circ f(z)$ (n times). Show that $F_n(z) \to 0$ as $n \to \infty$ uniformly on compact subsets of D.

[Hint: consider first the behavior of F_n on a small neighborhood of 0.]

4. Starting with the definition

"f is analytic on a set G if $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists for all $z_0 \in G$."

describe the sequence of intermediate results required to obtain the following theorem:

"Suppose f and g are both analytic on a connected open set G and there is a convergent sequence z_n with limit $z_\infty \in G$ such that $f(z_n) = g(z_n)$ for all n. Then $f = g$ on G."

[You do not need to prove any of the intermediate results, but you should give a brief indication of how each result is used to obtain the next one.]