Algebra Qualifying Exam -Spring 2010

1. Let $f(x)= x^6+3 \in Q[x]$. Show that the Galois group of f is S_3.

2. (a) Let G be a group of order pqr, where $p<q<r$ are primes. Show that G contains a normal subgroup of index p.
(b) Determine up to isomorphism all groups of order $3 \cdot 7 \cdot 13$.

3. Let R be a commutative Noetherian ring, and let I, J, and K be ideals of R. We say I is irreducible if $I = J \cap K$ ® $I = J$, or $I = K$.
(a) Show that every ideal of R is a finite intersection of irreducible ideals.
(b) Show that every irreducible ideal is primary. (An ideal I of R is primary if $R/I \neq 0$, and every zero-divisor in R/I is nilpotent.)

4. Let A be a finite-dimensional algebra ever a field K, such that for every $a \in A$, $a^7= a$. Show that A is a direct product (sum?) of fields. Which fields can arise?

5. Let G and H be finitely generated abelian groups such that $G \otimes_Z H = 0$. Show that G and H are finite and have relatively prime orders.

6. Let S and T be diagonalizable endomorphisms of a finite dimensional complex vector space. If S and T commute show that they are polynomials in each other.

7. What are the prime ideals of Z[x]? What are the maximal ideals? Carefully explain your answers.