ALGEBRA QUALIFYING EXAM FALL 2011

Work all of the problems. Justify the statements in your solutions by reference to specific results, as appropriate. Partial credit is awarded for partial solutions. The set of integers is $\boldsymbol{Z}$, the set of rational numbers is $\boldsymbol{Q}$, and set of the complex numbers is $\boldsymbol{C}$.

1. Let I and J be ideals of $R = \boldsymbol{C}[x_1, x_2, \ldots, x_n]$ that define the same variety of $\boldsymbol{C}^n$. Show that for any $x \in (I + J)/I$ there is $m = m(x) > 0$ with $x^m = 0_{R/I}$. Show there is an integer $M > 0$ so that for any $y_1, y_2, \ldots, y_M \in (I + J)/I$, $y_1 y_2 \cdots y_m = 0_{R/I}$.

2. If $K \subseteq L$ are finite fields with $|K| = p^n$ and $[L : K] = m$ then show that for each $1 \leq t < nm$, any $a \in L - K$ has a p^t–th root in L. When $m = 3$, show that every $b \in K$ has a cube root in L.

3. Let F be an algebraically closed field and A an F-algebra with $\dim_F A = n$. If every element of A is either nilpotent or invertible, show that the set of nilpotent elements of A is an ideal M of A, that M is the unique maximal ideal of A, and that $\dim_F M = n - 1$.

4. Let M be a finitely generated $F[x]$ module, for F a field.
   i) Show that if $f(x)m = 0$ for $f(x) \neq 0$ forces $m = 0$, then M is a projective $F[x]$ module.
   ii) If H is an $F[x]$ submodule of M show that $M = H \oplus K$ for a submodule K of M if and only if: $f(x)m \in H$ for $f(x) \neq 0$ implies that $m \in H$.

5. Up to isomorphism, describe the possible structures of any group of order $987 = 3 \cdot 7 \cdot 47$.

6. Let $R = \boldsymbol{Z}[x_1, x_2, \ldots x_n, \ldots]$ and let $\{f_i(X) \mid i \geq 1\} \subseteq R$ satisfy
   $f_1(X)R \subseteq f_2(X)R \subseteq \cdots \subseteq f_t(X)R \subseteq \cdots$. Show $f_s(X)R = f_m(X)R$ for some m and all $s \geq m$.

7. Let U be the set of all n-th roots of unity in $\boldsymbol{C}$, for all $n \geq 3$, and set $F = \boldsymbol{Q}(U)$. For primes $p_1 < \cdots < p_k$ and nonzero $a_1, \ldots, a_k \in \boldsymbol{Q}$, set $M = F(a_1^{1/p_1}, \ldots, a_k^{1/p_k}) \subseteq \boldsymbol{C}$. Show that M is Galois over F with a cyclic Galois group. For any subfield $F \subseteq L \subseteq M$, show that there is a subset T of $\{a_j^{1/p_j}\}$ so that $L = F(T)$.