ALGEBRA QUALIFYING EXAM FALL 2010

Do all six problems. Each problem is worth 4 points and partial credit may be awarded.

1. Use Sylow's Theorems to show that any group of order $(99^2 - 4)^3$ is solvable.

2. For any finite group G and positive integer m, let $n_G(m)$ be the number of elements g of G that satisfy $g^m = e_G$. If A and B are finite abelian groups so that $n_A(m) = n_B(m)$ for all m, show that as groups $A \cong B$.

3. If $g(x) = x^5 + 2 \in \boldsymbol{Q}[x]$, for $\boldsymbol{Q}$ the field of rational numbers, compute the Galois group of a splitting field L over $\boldsymbol{Q}$ of $g(x)$. How many subfields of L containing $\boldsymbol{Q}$ are Galois over $\boldsymbol{Q}$?

4. Let P be a minimal prime ideal in the commutative ring R with 1; that is, if Q is a prime ideal in R and if $Q \subseteq P$, then $Q = P$. Show that each $x \in P$ is a zero divisor in R.

5. Set $R = \boldsymbol{C}[x_1, \ldots, x_n]$ with $n \geq 3$ and $\boldsymbol{C}$ the field of complex numbers. For any subset $S \subseteq R$, let $\mathcal{V}(S) = \{\alpha \in \boldsymbol{C}^n \mid g(\alpha) = 0 \text{ for all } g \in S\}$. Consider the ideal I of R defined by $I = (x_1 \cdots x_{n-1} - x_n,\ x_1 \cdots x_{n-2}x_n - x_{n-2}, \ldots, x_2 \cdots x_n - x_1)$, so the generators of I are obtained by subtracting each x_j from the product of the others. Show that there are fixed positive integers s and t so that for each $0 \leq i \leq n$, $(x_i^s - x_i)^t \in I$. (Hint: Consider the product of the generators of I.)

6. Let R be a right artinian algebra over an algebraically closed field F. Show that R is algebraic over F of bounded degree. That is, show there is a fixed positive integer m so that for any $r \in R$ there is a nonzero $g_r(x) \in F[x]$ with $g_r(r) = 0$ and with $\deg g \leq m$.