ALGEBRA QUALIFYING EXAM, Fall 2009

Notation: $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, and $\mathbb{F}_p$ the field with p elements, for p a prime.

1. Determine up to isomorphism all groups of order $1005 = 3 \cdot 5 \cdot 67$.

2. (a) Let G be a group of order $2^m k$, where k is odd. Prove that if G contains an element of order 2^m, then the set of all elements of odd order in G is a (normal) subgroup of G.
(Hint: consider the action of G on itself by left multiplication Φ_L, and then consider the structure of the permutations $\Phi_L(x)$, for $x \in G$.)
(b) Conclude from (a) that a finite simple group of even order must have order divisible by 4.

3. Give a brief argument or a counterexample for each statement:
(a) $x^{2^n} + 1 \in \mathbb{Q}[x]$ is irreducible for all positive integers n;
(b) Any splitting field for $x^{13} - 1 \in \mathbb{F}_3[x]$ has 3^{12} elements. (c) $\mathcal{G}al(L/\mathbb{Q})$ for L a splitting field over $\mathbb{Q}$ of $x^5 - 2 \in \mathbb{Q}[x]$ has a normal 5-Sylow subgroup.

4. Let A denote the commutative ring $\mathbb{R}[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2 + 1)$.
(a) Prove that A is a Noetherian domain.
(b) Give an infinite family of prime ideals of A that are not maximal.

5. Let $R = \mathbb{C}[x_1, \ldots, x_n]$, let $A = [a_{ij}] \in M_n(\mathbb{C})$, and choose $b_1, \ldots, b_n \in \mathbb{C}$.
For each $i = 1, \ldots, n$, set $L_i = a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n - b_i \in R$, and consider the ideal $I = (L_1, \ldots, L_n) \subseteq R$.
Prove that R/I is finite-dimensional $\iff$ the matrix A is invertible in $M_n(\mathbb{C})$.

6. Let $R = K[x]$, for K a field, and let M be a finitely-generated torsion module over R. Prove that M is a finite-dimensional K-module.

7. Let G be a finite group and K a field, and consider the group algebra $R = KG$ (that is, R is a K-vector space with basis $\{g \in G\}$, and multiplication determined by the group product $g \cdot H$, for $g, h \in G$).
If G is the dihedral group of order 8, find the dimensions of all of the simple (left) modules for $R = \mathbb{F}_5 G$.
(Hint: remember that KG always has the "trivial representation" $V_0 = Kv$, such that for any $g \in G$, $a \in K$, $ag \cdot v = av$.)