ALGEBRA EXAM FEBRUARY 2011

1. Let G be a finite group with a cyclic Sylow 2-subgroup S.

(a) Show that any element of odd order in $N_G(S)$ centralizes S.

(b) Show that $N_G(S) = C_G(S)$.

(c) Give an example to show that (a) can fail if S is abelian.

2. Let G be a finite group with a cyclic Sylow 2-subgroup $S \neq 1$.

(a) Let $\rho : G \to S_n$ be the regular representation with $n = |G|$. Show that $\rho(G)$ is not contained in A_n.

(b) Show that G has a normal subgroup of index 2.

(c) Show that the set of elements of odd order in G form a normal subgroup N and $G = NS$.

3. For a group G and p a prime let $G(p) = \{g \in G | g^p = 1\}$.

(a) Show that if G is Abelian, then $G(p)$ is a subgroup of G. Give an example to show that $G(p)$ need not be a subgroup in general.

(b) Let G, H be finitely generated Abelian groups with $G/G(p) \cong H/H(p)$ and $G/G(q) \cong H/H(q)$ for different primes p, q. Show that $G \cong H$.

4. Let R be a prime ring with only finitely many right ideals.

(a) Show that R is a simple ring.

(b) Prove that either R is finite or R is a division ring.

5. Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and let J be a nonzero proper ideal of R. Let $A = A(X), B = B(X) \in M_r(R)$ and assume that $\det(A)$ is a product of distinct monic irreducible polynomials in R. Assume that for each $\alpha = (a_1, \ldots, a_n) \in \mathbb{C}_n$, $B(\alpha) \in M_r(\mathbb{C})$ invertible implies that $A(\alpha)$ is invertible. Show that $\det(A)$ divides $\det(B)$ in R.

6. Let L be a splitting field over $\mathbb{Q}$ for $p(x) = x^{10} + 3x^5 + 1$. Let $G = \mathrm{Gal}(L/\mathbb{Q})$.

(a) Show that G has a normal subgroup of index 2.

(b) Show that 4 divides $|G|$.

(c) Show that G is solvable.