Probability (507A) Graduate Exam

Fall 2011

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1. Let X_n, $n \geq 1$, and X be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Give the definitions for "$X_n \to X$ almost surely" and "$X_n \to X$ in L^1".

   (b) Give examples to show that (i) convergence almost surely does not imply convergence in L^1, and that (ii) convergence in L^1 does not imply convergence almost surely.

   (c) Prove that if $\sum_{n=1}^{\infty} \mathbb{E}|X_n - X| < \infty$ then $X_n \to X$ almost surely.

2. Let $X_1, X_2, \ldots$ be i.i.d. random variables uniformly distributed on $(0,1)$. Prove that

$$P\left\{\limsup_{n\to\infty}\left(\frac{-\log X_n}{\log\ n}\right) = 1\right\} = 1.$$

3. (a) Suppose that X and Y are each uniformly distributed on $(0,1)$, with $X+Y$ constant. In the base 2 expansions of X and of Y, determine how the ith bit for X relates to the ith bit for Y

   (b) Show that it is possible to have X, Y, Z each uniformly distributed on $(0,1)$, with $X+Y+Z$ constant. That is, give an explicit construction, or description, of the joint distribution of (X,Y,Z). [Hint: think about the base 3 expansion of a number in $(0,1)$.]