MATH 507a **QUALIFYING EXAM** **September 21, 2010**

Solve all four problems. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

Problem 1. Let $X_1, X_2, \ldots$ be a sequence of (not necessarily independent and not necessarily identically distributed) real random variables defined on a common probability space. Suppose that $\mathbb{E}(X_n^2) \leq 1$ for all $n \geq 1$. Does the sequence $X_n/n,\ n \geq 1$, necessarily converge almost surely to zero? Give a proof or a counterexample.

Problem 2. Let $X_1, X_2, \ldots$ be iid with density

$$f(x) = \begin{cases} 0, & \text{if } |x| \leq 1 \\ |x|^{-3}, & \text{if } |x| > 1. \end{cases}$$

Prove that

$$(n \log n)^{-\frac{1}{2}} \sum_{i=1}^{n} X_i \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

(that is, the expression on the left converges in distribution to a Gaussian random variable with mean zero and variance σ^2) and determine the value of σ^2.

Suggestion: Truncate $X_i,\ i = 1, \ldots, n$, at $\pm\sqrt{n \log n}$ and use the Central Limit Theorem for triangular arrays.

Problem 3. Let $X_k,\ k \geq 1$, be iid random variables such that

$$\limsup_{n \to \infty} \frac{X_n}{n} < \infty$$

with probability one. Show that

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} X_k}{n} < \infty$$

with probability one.

Suggestion: Apply the Law of Large Numbers to the sequence $\max(X_k, 0),\ k \geq 1$.

Problem 4. Let X and Y be independent random variables such that $\mathbb{E}|X + Y| < \infty$. Is it true that $\mathbb{E}|X| < \infty$? Give a proof or a counterexample.