MATH 507a **Screening Exam** **Fall, 2009**

Last Name: ____________________ **First Name:**____________________

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables uniformly distributed on $(0,1)$. Prove that

$$\mathbf{P}\{\limsup_{n\to\infty}\left(\frac{-\log X_n}{\log\ n}\right) = 1\} = 1.$$

2. Assume $X_1, X_2, \ldots$ are independent with

$$\mathbf{P}(X_n = n^{-\alpha}) = \mathbf{P}(X_n = -n^{-\alpha}) = \frac{1}{2}.$$

For what α does the series $\sum_n X_n$ converge a.s.? For what α does the series $\sum_n |X_n|$ converge a.s.?

3. Let U_1, U_2, ..., be an i.i.d. sequence of uniform random variables on $[0,1]$. Define a sequence of random variables $\{V_n\}$ recursively as follows:

$$V_1 = U_1, \qquad V_n = \begin{cases} 2V_{n-1}U_n & \text{if } V_{n-1} \in [0, \frac{1}{2}], \\ (2V_{n-1} - 1)U_n & \text{if } V_{n-1} \in [\frac{1}{2}, 1]. \end{cases}$$

(i) Show that, V_{n-1} and U_n are independent, for all $n \geq 2$;

(ii) $\mathbf{E}[V_n|V_{n-1}] = V_{n-1} - \frac{1}{2}\mathbf{1}_{\{1/2\leq V_{n-1}\leq 1\}}$, where $\mathbf{1}_{\{1/2\leq V_{n-1}\leq 1\}}$ is the indicator function of the set $\{1/2 \leq V_{n-1} \leq 1\}$;

(iii) Show that $\mathbf{P}(V_{n-1} < 1/2) \to a$, for some $a \in [0,1]$, as $n \to \infty$. Determine the number a.